ORDERS OF ELEMENTS IN THE EXTENSION OF THE SPECIAL LINEAR GROUP BY THE INVERSE TRANSPOSE INVOLUTION

M.A. Grechkoseeva

Sobolev Institute of Mathematics, 4 Koptyuga av., 630090, Novosibirsk, Russia
grechkoseeva@gmail.com

If G is a finite group, then we refer to the set of the orders of elements of G as the spectrum of G and denote this set by $\omega(G)$. Groups whose spectra coincide are said to be isospectral. Recently, the following assertion known as Mazurov's conjecture was established: if L is a finite simple sporadic group, or alternating group, or exceptional group of Lie type, other than J_2, A_6, A_{10} and ${}^3D_4(2)$, or if L is a finite simple classical group of dimension larger than 60, and G is a finite group isospectral to L, then up to isomorphism $L \leqslant G \leqslant \operatorname{Aut} L$ (see [1]).

A natural question arising in this context is when exactly $\omega(G) = \omega(L)$ provided that L is a finite nonabelian simple group and $L < G \leqslant \operatorname{Aut} L$ (cf. [2, Question 17.36]). The answer is known for sporadic and alternating groups, and is of prime interest for groups of Lie type. Every finite group of Lie type can be realized as $\overline{G}_\sigma = C_{\overline{G}}(\sigma)$ for some suitable simple linear algebraic group $\overline{G}$ and surjective endomorphism σ of $\overline{G}$. The spectra of certain extensions of $\overline{G}_\sigma$ can be computed using the following lemma due to Zavarnitsine [3, Proposition 13].

Lemma 1 (Zavarnitsine). *Let $\overline{G}$ be a connected linear algebraic group over an algebraically closed field of a positive characteristic. Let σ be a surjective endomorphism of $\overline{G}$ and denote $\overline{G}_k = C_{\overline{G}}(\sigma^k)$. If G_k is finite for some k, then σ is an automorphism of G_k and $\omega(G_k\sigma) = k\cdot\omega(G_1)$, where $G_k\sigma$ is a coset in $G_k \rtimes \langle\sigma\rangle$.*

Lemma 1 is a powerfull tool which allows one to handle extensions by diagonal and field automorphisms, but it cannot be applied to the extension of $PSL_n(q)$ by the involutory graph automorphism, or equivalently, by the inverse transpose automorphism. Recall that the inverse transpose automorphism of $GL_n(q)$ is the automorphism τ acting by $g^\tau = (g^\top)^{-1}$, where $g^\top$ denotes the transpose of g. Calculating the spectrum of $G = PSL_n(q) \rtimes \langle\tau\rangle$ is finding the orders of the elements of the coset $PSL_n(q)\tau$. Since $(g\tau)^2 = gg^\tau$, the latter problem is closely related to the equation $h = gg^\tau$ where h is a given element of $GL_n(q)$ and $g \in GL_n(q)$. This equation has been exhaustively studied by Fulman and Guralnick in [4]. Starting from their work, we first determine for what $h \in SL_n(q)$ there is $g \in SL_n(q)$ such that $gg^\tau = h$ and then resolve the question of isospectrality.

Theorem 1. *Let n and q be odd, $L = PSL_n(q)$, and let τ be the inverse transpose automorphism of L. Then $\omega(L\tau) = 2\cdot\omega(Sp_{n-1}(q))$. If q is a power of a prime p and $G = L\rtimes\langle\tau\rangle$, then $\omega(G) = \omega(L)$ unless one of the following holds:*

(1) *$q \equiv -1 \pmod 4$, $n = 2 + p^{k-1}$ for some $k \geqslant 1$, and $4p^k \in \omega(G) \setminus \omega(L)$;*

(2) *$n = 2^k + 1$ for some $k \geqslant 1$, $(n, q-1) \neq 1$, and $2(q^{(n-1)/2} - 1) \in \omega(G) \setminus \omega(L)$.*

Theorem 2. *Let $n \geqslant 4$ be even, q be a power of an odd prime p, $L = PSL_n(q)$, τ be the inverse transpose automorphism of L, and let $G = L \rtimes \langle\tau\rangle$. Then $\omega(G)$ is the joint of $\omega(L)$ and the set of all divisors of the following numbers:*

(i) *$2(q^{n/2} \pm 1)/(4, q^{n/2} \pm 1)$;*

(ii) *$2[q^{n_1} - \varepsilon_1, q^{n_2} - \varepsilon_2]/\delta$, where $2(n_1 + n_2) = n$, $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$, $\delta = 2$ if $(q^{n_1} - \varepsilon_1)_2 = (q^{n_2} - \varepsilon_2)_2$, and $\delta = 1$ otherwise;*

(iii) *$2p^k$ if $n = 1 + p^{k-1}$, $k \geqslant 2$.*

Furthermore, $\omega(G) = \omega(L)$ unless one of the following holds:

(1) *$q \equiv 1 \pmod 4$, $(n)_2 \leqslant (q-1)_2$, and $q^{n/2} + 1 \in \omega(G) \setminus \omega(L)$;*

(2) *$n = 1 + p^{k-1}$, $k \geqslant 2$, and $2p^k \in \omega(G) \setminus \omega(L)$;*

(3) *$(n, q-1)_{2'} \neq 1$, $(n)_{2'} > 3$, and $\omega(G) \setminus \omega(L)$ contains $2[q^{n_1} - 1, q^{n_2} + 1]$, where $n_1 = (n)_2$, $n_2 = n/2 - (n)_2$.*

In the above theorems, $(m)_2$ denotes the highest power of 2 dividing a positive integer m and $(m)_{2'}$ denotes $m/(m)_2$.

Observe that similar results can be derived for unitary groups since there is a one-to-one correspondence between the conjugacy classes in the coset $PSL_n(q)\tau$ and those in the coset $PSU_n(q)\tau$ (under some proper definition of $GU_n(q)$), and this correspondence preserves the order of the elements in a conjugacy class [5, Section 2].

References

1. Grechkoseeva M.A., Vasil'ev A.V. On the structure of finite groups isospectral to finite simple groups // J. Group Theory. DOI: 10.1515/jgth-2015-0019.

2. *The Kourovka notebook. Unsolved problems in group theory.* 17th edition. Institute of Mathematics, Novosibirsk, 2010.

3. Zavarnitsine A.V. Recognition of the simple groups $U_3(q)$ by element orders // Algebra Logic. 2006. V. 45. No. 2. P. 106–116.

4. Fulman J., Guralnick R. Conjugacy class properties of the extension of $GL(n, q)$ generated by the inverse transpose involution // J. Algebra. 2004. V. 275. No. 1. P. 356–396.

5. Gow R., Vinroot C.R. Extending real-valued characters of finite general linear and unitary groups on elements related to regular unipotents // J. Group Theory. 2008. V. 11. P. 299–331.